



NON-LINEAR NORMAL MODES AND NON-PARAMETRIC SYSTEM IDENTIFICATION OF NON-LINEAR OSCILLATORS

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The Karhunen–Loeve (K–L) decomposition procedure is applied to a system of coupled cantilever beams with non-linear grounding stiffnesses and a system of non-linearly coupled rods. The former system possesses localized non-linear normal modes (NNMs) for certain values of the coupling parameters and has been studied in the literature using various asymptotic techniques. In this work, the K–L method is used to locate the regions of such localized motions. The method yields orthogonal modes that best approximate the spatial behaviour of the beams. In order to apply this method simultaneous time series of the displacements at several points of the system are required. These measurements are obtained by a direct numerical integration of the governing partial differential equations, using the assumed modes method. A two-point correlation matrix is constructed using the measured time-series data, and its eigenvectors represent the dominant K–L modes of the system; the corresponding eigenvalues give an estimate of the participations (energies) of these modes in the dynamics. These participations are used to estimate the dimensionality of the system and to identify regions of localized motion in the coupling parameter space. The same approach is applied to a system of non-linearly coupled rods. Through the comparison of system response reconstructions of the responses using a simple K–L mode and a number of physical modes, it is shown that the K–L modes can be used to create lower-order models that can accurately capture the dynamics of the original system.

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1. INTRODUCTION

In recent works it has been shown that linear or non-linear weakly coupled structures can be designed to possess localized non-linear normal modes (NNMs). These NNMs represent spatially confined free vibrations where all material points of a structure vibrate synchronously. Localization in weakly coupled linear periodic systems occur only when weak perturbations of the periodicity (structural disorders) exist [1–3]; however in a non-linear setting no such requirements exist [4–7]. It has also been shown that structures with localized modes also possess passive motion confinement properties [7, 8]. In [9] the localization and motion confinement are experimentally investigated in a system of coupled non-linear beams with active non-linear grounding stiffnesses. In another related work [10], it is shown that when the grounding non-linearities are of impacting nature, the motion confinement is further enhanced.

In the current investigation, the method of Karhunen–Loeve (K–L) decomposition is used to identify localized motions and to study the low-dimensional dynamic models in a system of coupled beams with non-linear (cubic) grounding stiffness and a system of non-linearity coupled rods. The systems under consideration are closely related to those presented in [9, 10]. The use of the Karhunen–Loeve (K–L) transform [11] is of great value

in non-linear settings where traditional linear system identification techniques such as least-squares identification, modal testing and power spectrum analysis cannot be applied. This is especially the case for non-linear engineering structures. The Karhunen–Loeve decomposition is used to obtain low-dimensional dynamic models of distributed parameter systems, by computing orthogonal eigenfunctions derived by post-processing experimental or numerical data of the system response. These eigenfunctions are optimal in the sense that fewer K–L modes are needed to account for the same amount of vibrational energy, compared to modes resulting from application of standard Galerkin or Rayleigh–Ritz procedures [12]. This technique can conveniently treat non-linear distributed parameter systems defined on irregular domains to yield discretized systems with only a few degrees of freedom, which can then be used to reconstruct the dynamical response.

The Karhunen–Loeve analysis has additional distinctive advantages. The modes obtained from the K–L decomposition for a certain set of system parameters, can be used to reconstruct the response of a system whose parameters are different from the original system. The key advantage of this method lies in the fact that it can be applied, not only to conservative systems, but also to dissipative ones, and provides information about coherent spatial structures in the dynamics as well as quantification of the energies contained in them. Hence, this method could be a valuable tool in the analysis and system identification of the dynamics of practical engineering structures. This method had been applied successfully in the fields of fluid dynamics, thermal problems and signal processing. In [13] the K–L method has been applied to a turbulent thermal convective system and low-order models were created to study the dynamics of the thermal behavior. In [14] the modes of a reaction–diffusion chemical process are captured by means of the K–L method and the dynamical behaviour ascertained. The snapshot method is used by Rajaei *et al.* [15] to describe the dynamics of the coherent structure of a weakly perturbed free shear layer. Sirovich and Kirby, [16] use the K–L method of snapshots to capture the dynamical structure of a two-dimensional axisymmetric jet. Sirovich employs the K–L method in different fluid dynamical contexts as reported in [12, 16–18] in [19], Park and Cho examine the efficiency of the K–L method for the control of distributed parameter systems. The notable work in structural mechanics are those by Mari and Glangeaud [20] and Cusumano *et al.* [21]. In the latter the dimensionality of the dynamics of an impacting beam is studied by means of traditional time-delay techniques and K–L decomposition. The energy transfer between K–L modes is also studied experimentally, in that work. In [22] Bayly and Virgin apply K–L decomposition to study the stability of the periodic motions of a forced spring–pendulum system.

In this work, the time responses of the beams at several spatial locations of a non-linear system are obtained numerically by an assumed mode analysis. These time traces are then used in a K–L analysis, to yield the dominant spatial structures and their energies. These energies are used to identify regions of localized motion and to determine the dimensionality of the system. The obtained K–L modes are then used to create lower-dimensional models with which the system dynamics is reconstructed and compared with the actual responses.

2. SYSTEM OF COUPLED BEAMS

2.1. THEORETICAL MODELLING

The system under consideration is depicted in Fig. 1. It consists of two beams which are coupled by means of a linear spring and the beams are connected to the ground by means of

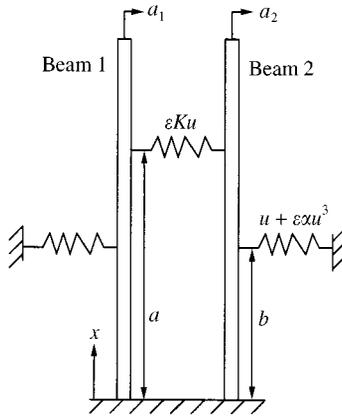


Figure 1. Schematic of the system under consideration.

cubic springs. Assuming linear Euler–Bernoulli theory for the beam vibrations, the governing Bernoulli partial differential equations (PDEs) are given by

$$m \frac{\partial^2 u_1}{\partial t^2} + EI \frac{\partial^4 u_1}{\partial x^4} = -\epsilon k \delta(x-a)[u_1(a,t) - u_2(a,t)] - \delta(x-b)[u_1(b,t) + \epsilon \alpha u_1^3(b,t)] + f_1(x,t) \quad (1)$$

$$m \frac{\partial^2 u_2}{\partial t^2} + EI \frac{\partial^4 u_2}{\partial x^4} = \epsilon k \delta(x-a)[u_1(a,t) - u_2(a,t)] - \delta(x-b)[u_2(b,t) + \epsilon \alpha u_2^3(b,t)] + f_2(x,t) \quad (2)$$

where $x = a$ denotes the position of linear coupling, $x = b$ the position of the non-linear grounding springs, ϵk the linear coupling stiffness, $\epsilon \alpha$ the non-linear coefficient of the grounded spring, m and EI , the uniform mass and elasticity distributions per unit length of the beams, $u_{1,2}(x, t)$ the transverse displacements of beams 1 and 2, respectively, and $f_{1,2}(x, t)$ the external forcing. The numerical values for the beams parameters are: $EI = 57.76 \text{ Nm}^2$, $m = 0.753 \text{ kg/m}$, $a = 0.5 \text{ m}$, $b = 0.306 \text{ m}$ and the length of the beams $L = 0.766 \text{ m}$. The set of equations (1) and (2) are solved by discretization by expressing the transverse displacements in the following series forms:

$$u_1(x, t) = \sum_{i=1}^{\infty} A_{1,i}(t) \varphi_i(x) \quad \text{and} \quad u_2(x, t) = \sum_{i=2}^{\infty} A_{2,i}(t) \varphi_i(x) \quad (3)$$

where $\varphi_i(x)$ represents the i th normalized linear cantilever mode, and $A_{p,i}(t)$, $p = 1, 2$, the amplitude of the i th mode of beam p . Employing the orthogonality conditions satisfied by the cantilever modes, the original partial differential equations of motion are replaced by the following set of non-linear ordinary differential equations which over the time evolution of the modal amplitudes:

$$\ddot{A}_{1,j}(t) + \omega_j^2 A_{1,j} = -\frac{\epsilon k \varphi_j(a)}{m C_j} \left\{ \sum_{i=1}^{\infty} [A_{1,i}(t) - A_{2,i}(t)] \varphi_i(a) \right\} - \frac{\varphi_j(b)}{m C_j} \left\{ \sum_{i=1}^{\infty} A_{1,i}(t) \varphi_i(b) + \epsilon \alpha \left[\sum_{i=1}^{\infty} A_{1,i}(t) \varphi_i(b) \right]^3 \right\}$$

$$+ \frac{1}{mC_j} \int_0^L f_1(x, t) \varphi_j(x) dx \quad (4)$$

$$\begin{aligned} \ddot{A}_{2,j}(t) + w_j^2 A_{2,j} = & - \frac{\varepsilon k \varphi_j(a)}{mC_j} \left\{ \sum_{i=1}^{\infty} [A_{1,j}(t) - A_{2,j}(t)] \varphi_i(a) \right\} \\ & - \frac{\varphi_j(b)}{mC_j} \left\{ \sum_{i=1}^{\infty} A_{2,j}(t) \varphi_i(b) + \varepsilon \alpha \left[\sum_{i=1}^{\infty} A_{2,i}(t) \varphi_i(b) \right]^3 \right\} \\ & + \frac{1}{mC_j} \int_0^L f_2(x, t) \varphi_j(x) dx \quad (5) \end{aligned}$$

where $j = 1, 2, \dots$, w_j^2 is the natural frequency squared of the j th linearized cantilever mode, and $C_j = \int_0^L \varphi_j^2(x) dx$ the normalization coefficient of the j th mode.

In the numerical computations, the infinite set of equations are truncated to a finite number of terms by considering only the first N modes in the series expressions [Equation (3)]. In a similar study [10], the first 3 modes performed well in the time-evolution studies of the modal amplitudes. The integration of equations (4) and (5) are performed using a fourth-order Runge–Kutta numerical scheme. The free vibrations of the beams are then studied when the beams are subject to the following initial conditions:

$$u_1(x, 0) = \frac{F_1 x^2}{6EI} (3L - x) \quad u_2(x, 0) = \frac{F_2 x^2}{6EI} (3L - x) \quad \text{and} \quad \frac{\partial u_{1,2}(x, 0)}{\partial t} = 0 \quad (6)$$

which correspond to zero velocity and an initial displacement of the beams using a force $F_{1,2}$ applied at the corresponding tips.

2.2. NON-LINEAR NORMAL-MODE LOCALIZATION

The important parameters that determine the dynamics of the system are εk and $\varepsilon \alpha$. For a given set of parameters the system exhibits markedly different types of behaviour depending on the initial conditions and the total energy (ρ) contained in the system. The different possible normal-mode free oscillations for the initial conditions that are considered here are depicted in Fig. 2. In Fig. 2(a) the beams undergo in-phase motions of the same magnitude, in Fig. 2(b) the motions of the beam are of same magnitude but antiphase to each other, in Fig. 2(c) there is a localization of beam 1 and it is antiphase to beam 2 and Fig. 2(d) antiphase localization in beam 2. These normal modes of vibration will be referred to as Types I, II, III and IV nonlinear normal modes, respectively. If the system has linear stiffnesses, only Type I and II modes can be obtained.

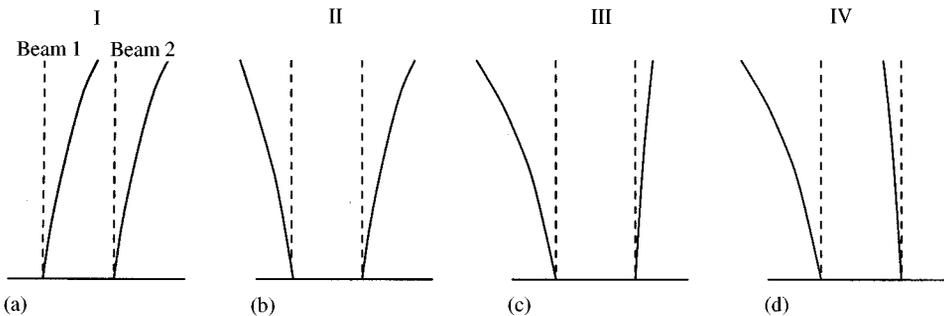


Figure 2. The different types of non-linear normal modes possible for the coupled beam oscillator.

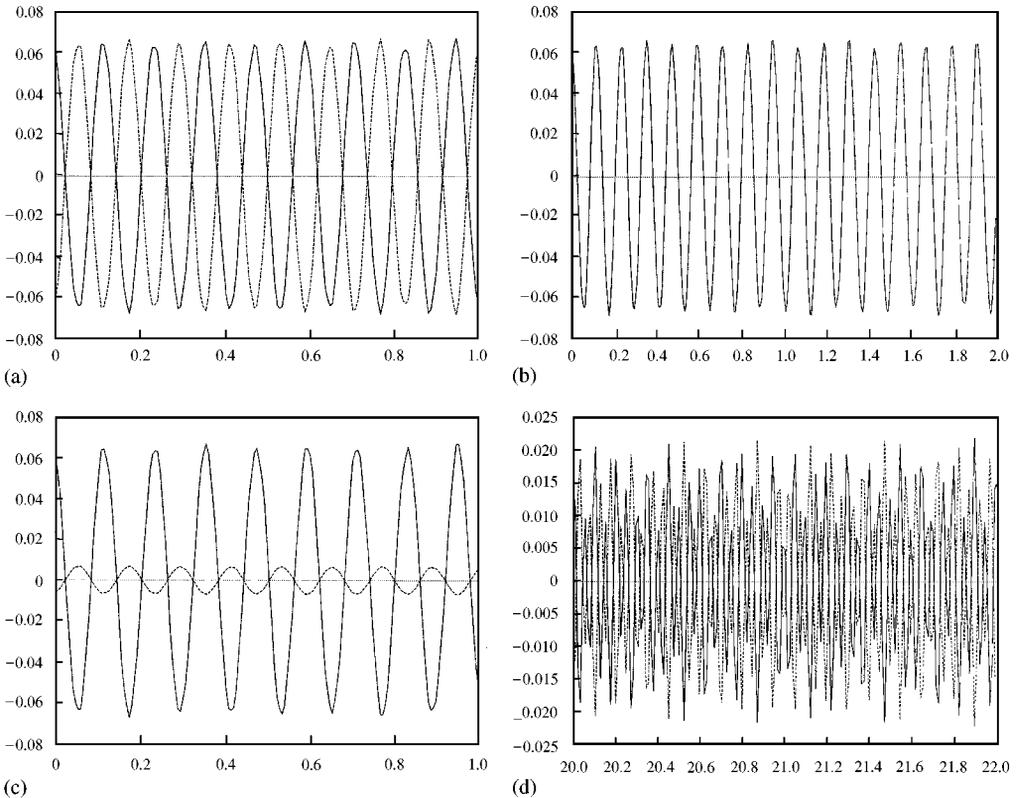


Figure 3. The displacements of the beams at $x = 0.766$ m for (a) $F_1/F_2 = -1$, $\varepsilon k = 10$, $\varepsilon \alpha = 1$, (b) $F_1/F_2 = 1$, $\varepsilon k = 10$, $\varepsilon \alpha = 1$, (c) $F_1/F_2 = -10$, $\varepsilon k = 0.1$, $\varepsilon \alpha = 1$, (d) $F_1/F_2 = -1$, $\varepsilon k = 1$, $\varepsilon \alpha = 100$: — beam 1; ... beam 2.

It has been observed that the system possesses two stable modes when the ratio $k/\alpha\rho^2$ is greater than a critical value. These stable modes correspond to Type I and Type II modes. However, if this ratio is lesser than the critical value, three stable modes and one unstable mode are found to exist. In this case the unstable mode is of Type II. The existence of these solutions (obtained by numerical integrations) are shown in Fig. 3. Figures 3(a), (b), (c) and (d) correspond to a stable Type II mode, a stable Type I mode, a stable localized Type III mode, and a unstable Type II mode, respectively.

When the system is forced it contains a combination of all the above modes. In a practical situation it is of interest to study the localization of a forced system, for applications in passive motion confinement. Localized motions occur when the dynamics of the beams are dominated by Type III or IV modes. However, the application of asymptotic techniques to study localization in a forced non-linear system becomes difficult. In such a situation, brute force numerical simulations have to be used to study the motion confinement properties of the system. Here we propose to use the K–L method to identify the dominant modes of the two beams and study localization properties. A brief introduction to the K–L procedure and the numerical scheme used are outlined in the next section.

2.3. NON-LINEAR LOCALIZATION ANALYSIS BY K–L DECOMPOSITION

The method of proper orthogonal decomposition (POD), which is also known as Karhunen–Loeve (K–L) decomposition is used to extract spatial information from a set of

time-series data available on the spatial domain. The detailed description of the method can be found in [11]. There are two methods of solving the K–L problem: the direct and snapshot method [23]. In this work the direct method is used. The displacements (u_p) of each beam are obtained at n locations at time instances t_k , $k = 1, \dots, N$. Using these displacements a time-averaged two-point correlation matrix (K) defined as follows is created:

$$K_p(x_i, x_j) = \frac{1}{N} \sum_{k=1}^N u_p(x_i, t_k) u_p(x_j, t_k) \quad (7)$$

where $x_i, i = 1, \dots, m$, are the evenly spaced points on the beam, and t_i are the discrete points in the time domain where the displacements are available. The K–L method requires the following integral eigenvalue problem to be solved [23] in order to extract the dominant modes:

$$\int_{x_1}^{x_m} K_p(x_i, x_j) \varphi(x_j) dx_j = \lambda \varphi(x_i) \quad (8)$$

where $\varphi_i, i = 1, \dots, n$, are the eigenfunctions (or the K–L modes) and λ_i represents the amount of energy stored in each of these modes. If λ_i are ordered in a decreasing fashion, the corresponding φ_i are in decreasing order of dominance. It should be noted that the K–L modes have no physical significance other than the fact that they best represent the spatial behaviour of the system. However, for an undamped and unforced system, the K–L modes are shown to converge to the physical modes of the system as the number of measurement points tends to infinity [24]. The method yields n eigenvalues and only the first p modes that satisfy the following relation are retained:

$$\frac{\sum_{i=1}^p \lambda_i}{\sum_{i=1}^n \lambda_i} \geq 0.99. \quad (9)$$

Hence p gives a dimensionality estimate of the beam. The above procedure is carried out for each of the beams separately. The i th K–L mode and energy of the j th beam are denoted by φ_i^j and λ_i^j respectively, where $i = 1, 2, \dots, n$ and $j = 1, 2$.

The K–L procedure is applied to the coupled beam oscillator with zero initial conditions and forcing $f_{1,2}$ given as follows:

$$\begin{aligned} f_1(x, t) &= F(t)(x^4 - x^3/2 - x^2/2 + x^5/100) \\ f_2(x, t) &= F(t)(x^4 + x^3/2 + x^2/100 - x^5) \\ F(t) &= 1000 \text{ for } 0 < t < 0.1 \text{ and } 0 \text{ otherwise} \end{aligned} \quad (10)$$

This forcing represents an almost impulse like forcing applied to each of the beams. The effect of the linear coupling stiffness k on the K–L modes and its energy, are studied for $\alpha = 10^6$ and $\varepsilon = 0.1$. The value of k is changed from 0 to 1000; $k = 0$ corresponds to no coupling between the beams and a large k value corresponds to strong coupling and hence the beams move together. The first K–L mode shapes for a few values of k are presented in Fig. 4 and it can be seen that the K–L mode shapes for the two beams are almost the same. However, the energies contained in each of them is different as shown in Table 1. For k values of 10, 50 and 100, the motion is confined to the second beam, while for a higher k value of 150, the motions of both the beams are comparable.

In order to compare how the non-linear effects enhance the motion confinement properties of the system, the energy ratio (λ_1^2/λ_1^1) of the first K–L modes of the two beams are plotted against the coupling stiffness value k , for the linear ($\alpha = 0$) and non-linear ($\alpha \neq 0$)

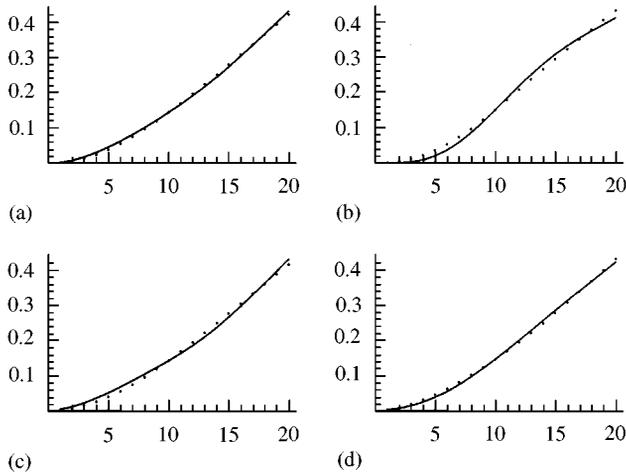


Figure 4. First K-L mode shapes of beam 1 (—) and beam 2 (···) when they are weakly coupled ($\varepsilon = 0.1$, $a = 10^6$) for (a) $k = 10$, (b) $k = 50$, (c) $k = 100$ and (d) $k = 150$.

TABLE 1
Energy of the first K-L modes of the two beams
for $\varepsilon = 0.1$, $\alpha = 10^6$

k	λ_1^1	λ_1^2
10	0.23102E-01	0.77925E-01
50	0.16703E-01	0.80181E-01
100	0.22117E-01	0.70314E-01
150	0.46976E-01	0.41124E-01

cases. This comparison is presented in Fig. 5. If the value of this ratio is greater than unity, the motion is confined to the second beam and hence localized motions occur. It can clearly be seen that in the range of $k \in [50, 100]$, there is a high degree of localized motion in the non-linear system. However, as the coupling stiffness is increased, the non-linear effects diminish and the behaviour of the linear and non-linear systems are almost the same. For very low k values, the applied forces $f_{1,2}$ dominate the responses as the beams are almost uncoupled and hence localization or non-localization is dictated by the forces.

2.4. RECONSTRUCTION OF DYNAMICAL RESPONSE USING K-L MODES

The energy stored in K-L modes gives the measure of the dimensionality of the system. The modes can be used to reconstruct the dynamics of the system using a reduced-order dynamical model. Hence instead of using several physical modes of the cantilever beam to discretize the governing PDEs [equations (4) and (5)], a few K-L modes can be used in the discretization process to create a lower-dimensional model. The lower-order model created is represented as follows:

$$\begin{aligned} \ddot{A}_{1,j}(t) \int_0^L \varphi_j^2(x) dx + EI \sum_{i=1}^p A_{1,i}(t) \int_0^L \varphi_0^{(4)} \varphi_i^{(4)} \varphi_j(x) dx \\ = -\frac{\varepsilon k}{m} \varphi_j(a) \left\{ \sum_{i=1}^p [A_{1,j}(t) \varphi_i(a) - A_{2,j}(t) \psi_i(a)] \right\} \end{aligned}$$

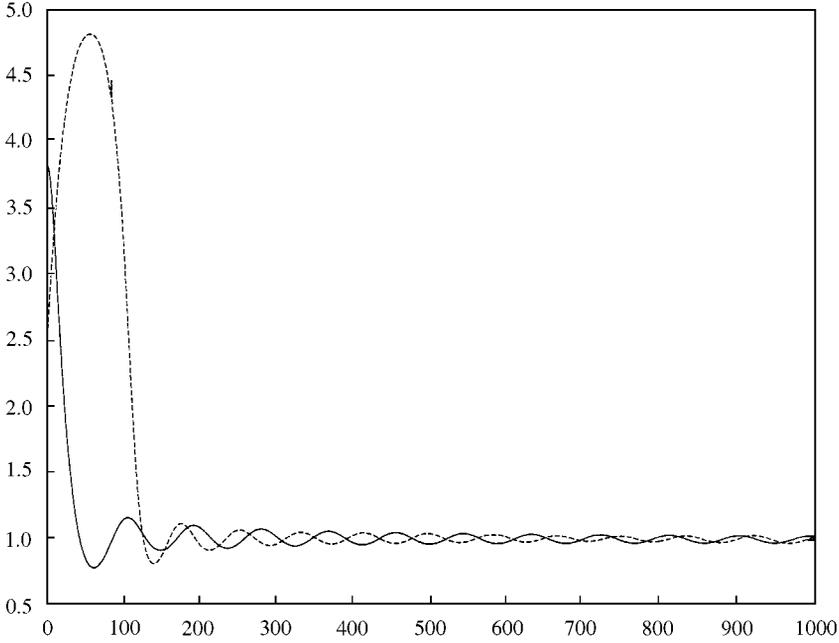


Figure 5. Ratio of energies of the first two K-L modes, $\varepsilon = 0.1$, without non-linearity $\alpha = 0$, with non-linearity $\alpha = 10^6$: — linear; ---- Nonlinear.

$$\begin{aligned}
 & -\frac{1}{m} \varphi_j(b) \left\{ \sum_{i=1}^p [A_{1,i}(t) \varphi_i(b) + \varepsilon \alpha \left[\sum_{i=1}^p [A_{1,i}(t) \varphi_i(b)]^3 \right] \right\} \\
 \ddot{A}_{2,i}(t) \int_0^L \psi_j^2(x) dx + EI \sum_{i=1}^p A_{2,i}(t) \int_0^L \psi_i^{(4)} \psi_j(x) dx \\
 & = \frac{\varepsilon k}{m} \psi_j(a) \left\{ \sum_{i=1}^p [A_{1,j}(t) \varphi_i(a) - A_{2,j}(t) \psi_i(a)] \right\} \\
 & - \frac{1}{m} \psi_j(b) \left\{ \sum_{i=1}^p [A_{2,i}(t) \psi_i(b) + \varepsilon \alpha \left[\sum_{i=1}^p [A_{2,i}(t) \psi_i(b)]^3 \right] \right\} \quad (11)
 \end{aligned}$$

where $\varphi_i(x)$ and $\psi_i(x)$ are the K-L modes of beams 1 and 2, respectively, and p stands for the number of K-L modes used to reconstruct the response of the system. The efficacy of the K-L modes for the reconstruction of the dynamical responses is illustrated by means of two different simulations. In the first, the linear coupling stiffness $\varepsilon k = 10^3$ and the grounding stiffness $\varepsilon \alpha = 10^6$. For this system at least two physical cantilever modes are required for capturing the system dynamics; however only one K-L mode is sufficient to reconstruct the response (cf. Figs 6(a) and (b)). The results of the second simulation for which $\varepsilon k = 10^3$ and $\varepsilon \alpha = 10^6$ is presented in Fig. 7 and the same trend is observed.

3. SYSTEM OF NON-LINEARLY COUPLED RODS

3.1. THEORETICAL MODELLING

The system of the non-linearly coupled rods is depicted in Fig. 8. The excitation force is applied at one end of the left rod and the right rod is connected to the ground

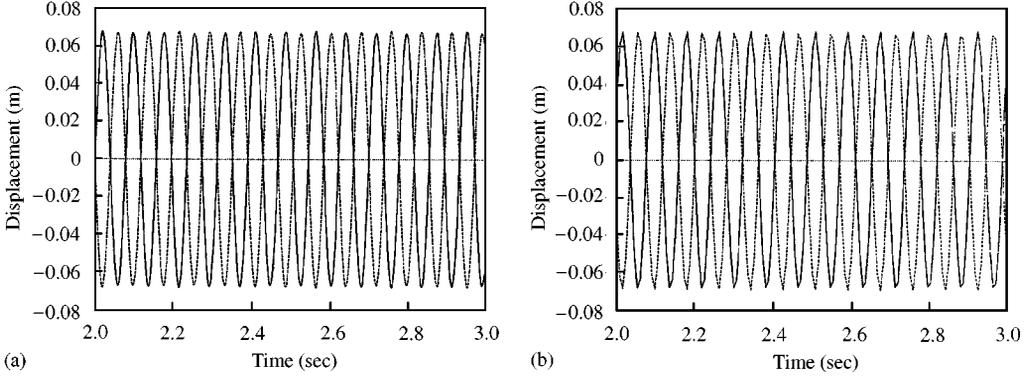


Figure 6. The comparison of the reconstructed beam response for a system for which $\epsilon k = 10^3$, $\epsilon x = 10^6$ using (a) three cantilever modes and (b) first K-L mode: — beam 1; \cdots beam 2.

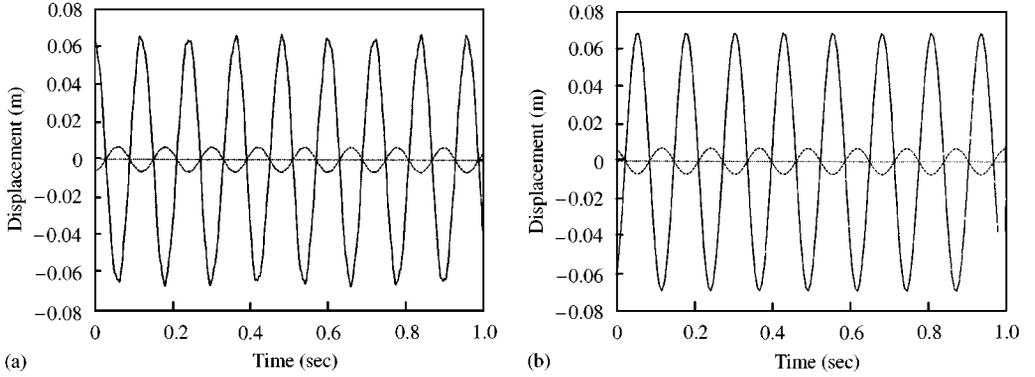


Figure 7. The comparison of the reconstructed beam response for a system for which $\epsilon k = 0.1$, $\epsilon x = 1$ using (a) three cantilever modes and (b) first K-L mode: — beam 1; \cdots beam 2.

by a linear spring. The governing partial differential equations are given by the following equations:

$$m \frac{\partial^2 v_1}{\partial t^2} - EA \frac{\partial^2 v_1}{\partial x^2} = -K_1 \delta(x-L)[v_2(0,t) - v_1(L,t) + (v_2(0,t) - v_1(L,t))^3] + F(x,t)\delta(x) \quad (12)$$

$$m \frac{\partial^2 v_2}{\partial t^2} - EA \frac{\partial^2 v_2}{\partial x^2} = -K_1 \delta(x-L)[v_2(0,t) - v_1(L,t) + (v_2(0,t) - v_1(L,t))^3] - K_2 v_2(L,t)\delta(x-L) \quad (13)$$

where K_1 is the non-linear coupling stiffness, K_2 the linear stiffness of the grounded spring, m and EA the uniform mass and elasticity distributions per unit length of the rods, $v_{1,2}(x,t)$ the longitudinal displacements of rods 1 and 2, respectively, L the length of each rod and $F(x,t)$ the external force. The numerical values for these parameters are, $EA = 4842.3068$ N, $m = 0.753021$ kg/m, and $L = 1.00$ m.

Employing the same discretization method as the beam system, the partial differential equations of motion are changed into a set of non-linear ordinary differential equations. Then by using a fourth-order Runge-Kutta numerical scheme to integrate the equations,

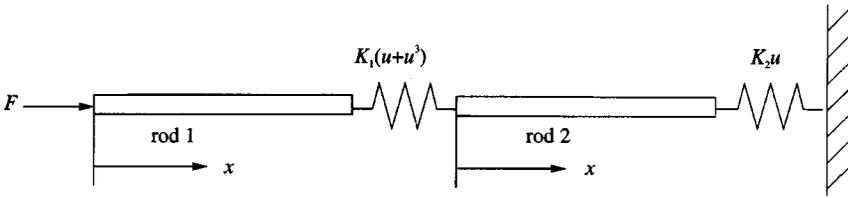


Figure 8. Schematic of the coupled system of rods.

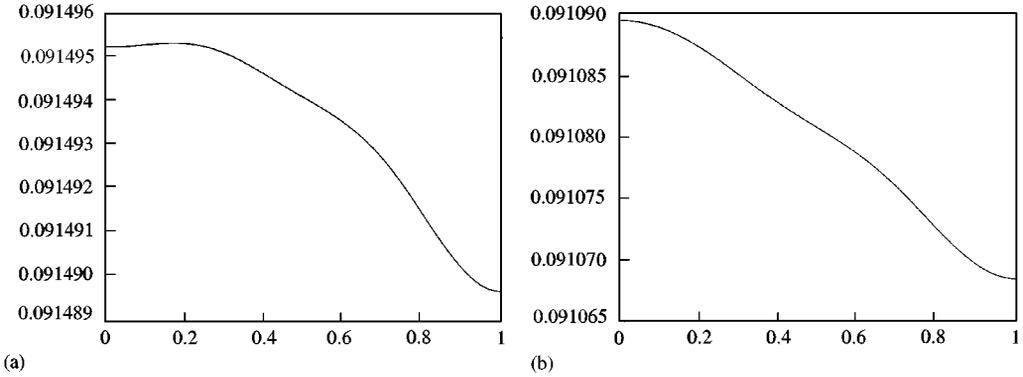


Figure 9. First K-L mode shapes of rod 1 (a) and rod 2, — newmode 1.dat; (b) when $K_1 = 500$, $K_2 = 5$, — newmode 2.dat.

the system response is obtained. In this case, five physical modes are used to study the rod system. The rods are initially at rest and are excited by the following force.

$$F(t) = 10 \sin(20t) \quad \text{for } 0 < t < 1 \text{ and } 0 \text{ otherwise.} \tag{14}$$

3.2. K-L DECOMPOSITION AND RECONSTRUCTION OF THE SYSTEM RESPONSE

The efficiency by which dynamic response can be simulated using K-L modes as opposed to normal physical modes is shown as follows. First the K-L modes of the system are found by application of the K-L decomposition method. The first K-L mode is depicted in Fig. 9, and it is important to note that the corresponding energy of the mode is 99.996% of the total energy of the system. Using this dominant mode the response of rods 1 and 2 are shown in Figs 10(c) and 10(d), respectively. Compare with Figs 10(a) and 10(b) which depict numerical results using five physical modes in the simulation. We see that a simple K-L mode captures the response predicted by five physical modes. When taking the alternative approach by using first 2 physical modes (include one rigid mode and the first non-rigid mode), the system response is shown in Figs 10(e) and 10(f). Clearly, this alternative approach fails to accurately describe the dynamics. So the first K-L mode captures almost the total energy of the system and it is sufficient to reconstruct the response.

4. REMARKS

The K-L method has great potential in application to diagnosis of non-linear systems. It is a non-linear, non-parametric system identification technique which can be applied to

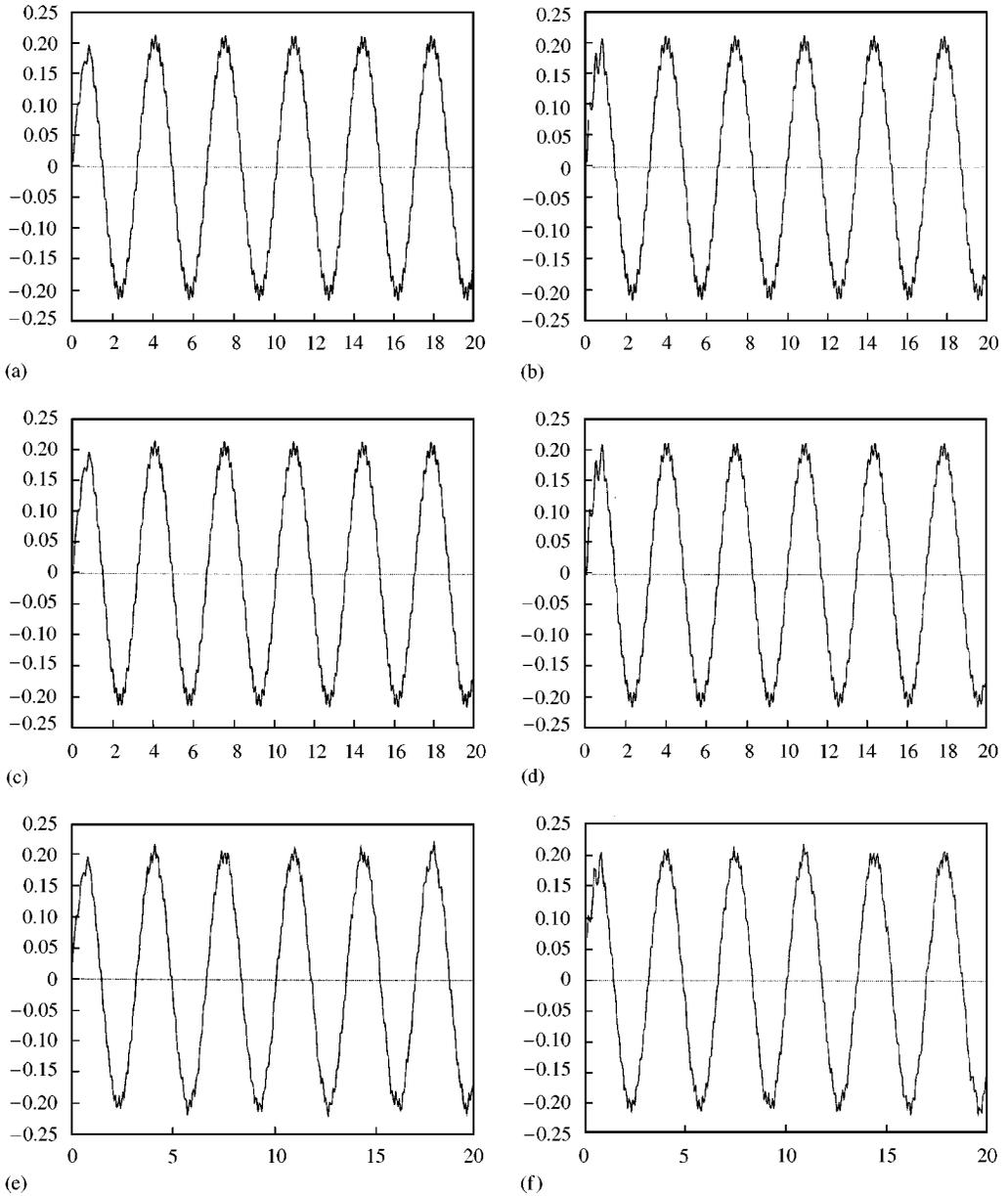


Figure 10. The comparison of the reconstructed rod responses for a system with $K_1 = 500$, $K_2 = 5$. (a) $x = L$ at rod 1 using five physical modes, — u1.dat; (b) $x = L$ at rod 2 using five physical modes, — u2.dat; (c) $x = L$ at rod 1 using the first K-L mode, — resu1.dat; (d) $x = L$ at rod 2 using the first K-L mode, — resu2.dat; (e) $x = L$ at rod 1 using two physical modes, — u1.dat; — u1.dat; (f) $x = L$ at rod 2 using two physical modes, — u2.dat.

many practical problems. The method can be used to create lower-order models which can capture system responses accurately. This could result in considerable savings in computational time if applied to a large-scale problem. In this work, it has been used to identify regions of localized motion, which is important in the design of passive vibration isolation structures. The method is not highly time consuming and hence can also be used for online fault detection.

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